A Fully Magnetizing Phase Transition

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Abstract

We analyze the Farey spin chain, a one dimensional spin system with effective interaction decaying like the squared inverse distance. Using a polymer model technique, we show that when the temperature is decreased below the (single) critical temperature $T_c = \frac{1}{2}$, the magnetization jumps from zero to one.

1 Introduction

Can a magnet keep its full mean magnetization $\langle m \rangle = 1$ up to the Curie temperature T_c and then loose it at one stroke? Definitely such a property would be different from the usual situation, where $\langle m \rangle$ continuously decreases to zero (though not being differentiable at T_c), or jumps discontinuously by an amount strictly less than the saturation value.

It has been proven [1, 2, 4, 5, 13, 15] that certain spin chains of long range ferromagnetic interaction exhibit a discontinuity of $\langle m \rangle$ at T_c, jumping from a value in the interval $(0, 1)$ to zero.

In one dimension such a phenomenon can only occur if the effective interaction between spins of distance d decays at most like d^{-2}, since there cannot be a phase transition for a decay rate of $d^{-\alpha}$ if $\alpha > 2$.

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However these examples do not exactly provide a positive answer to the question posed initially, since the jump of $\langle m \rangle$ at T_c is strictly smaller than one. Indeed for non-zero temperatures a mean magnetization $\langle m \rangle = \pm 1$ is only possible if the forces between the spins become so strong that one may legitimately ask whether this could endanger the existence of a thermodynamic limit.

In the present paper we show the contrary by considering the example of the Farey fraction spin chain. Similar to [8] the abstract polymer model formalism, is introduced in *Sect. 2*. Since the limit free energy coincides with the one of the number-theoretical spin chain (*Theorem 3*), the single phase transition (nonanalyticity of the free energy density) is situated at inverse temperature $\beta = 2$.

In *Sections 5* resp. *6* we consider the mean square magnetization in the regimes below resp. above the temperature. Whereas $\langle m^2 \rangle (\beta) = 1$ for low temperatures (*Theorem 5*), this quantity vanishes above T_c (*Theorem 7*).

We conjecture, and plan to prove, that the spin chain has exactly two extremal Gibbs measures in its low temperature phase, and one above T_c .

We also invoke a polymer model technique similar to the one developed in [8] to estimate the strength of the interaction.

2 The Model

In [10] the so-called *number-theoretical spin chain* was introduced, whose low-temperature partition function equals a quotient of Riemann zeta functions. In a series of subsequent papers (see [11] for a survey) this model was then analyzed further. In particular it was shown in [3] that a phase transition with a jump of m from one to zero occurs at $T_c = \frac{1}{2}$.

The number-theoretical spin chain shows an asymptotic decay of interactions which is exactly of the form d^{-2}, and the limit free energy density exists. The main motivation of its study lies in its connection with number theory, and more specifically in the hope that its ferromagnetic character together with a version of the Lee-Yang theorem could shed a light on the location of the zeroes of the Riemann zeta function.

¿From the statistical mechanics point of view it should, however, be said that it lacks the strict symmetries usually encountered in ferromagnets. It is neither fully translation invariant nor invariant under spin reversal, although both

symmetries are asymptotically present in the bulk.

In [9] the so-called *Farey fraction spin chain* was introduced as a spin system of statistical mechanics related to the Farey fractions in number theory. As we shall state below, this chain, which has strong relations with the one mentioned above, but a less direct number-theoretical interpretation of its partition function, has all relevant symmetries.

The definition of the Farey chain in [9] was based on functions

$$M_k : \mathbf{G}_k \to \mathrm{SL}(2,\mathbb{Z}) \qquad (k \geq 0)$$

on the additive group $\mathbf{G}_k := \{0,1\}^{\{1,\ldots,k\}}$, inductively defined by setting $M_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and for $k \geq 1$

$$M_k(\sigma) := A^{1-\sigma_k} B^{\sigma_k} M_{k-1}(\sigma_1, \ldots, \sigma_{k-1}) \qquad (\sigma \in \mathbf{G}_k), \tag{1}$$

with $A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B := A^t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The function

$$E_k := \ln(T_k) \quad \text{with} \quad T_k := \mathrm{Trace}(M_k) : \mathbf{G}_k \to \mathbb{N}$$

was interpreted as the *energy function* of a spin chain with k spins with values $\sigma_1, \ldots, \sigma_k$.

Then by discrete Fourier transformation

$$(\mathcal{F}_k f)(t) := 2^{-k} \sum_{\sigma \in \mathbf{G}_k} f(\sigma) \cdot (-1)^{\sigma \cdot t} \qquad (t \in \mathbf{G}_k) \tag{2}$$

the energy function has the form

$$E_k(\sigma) = -\sum_{t \in \mathbf{G}_k} J_k(t)(-1)^{\sigma \cdot t} \qquad (\sigma \in \mathbf{G}_k)$$

with the so-called *interaction coefficients*

$$J_k(t) := -(\mathcal{F}_k E_k)(t) \qquad (t \in \mathbf{G}_k).$$

The ‘lattice gas’ spin values $\sigma_i = 0, 1$ are used here for convenience. The mean magnetization

$$m_k := \frac{1}{k} \sum_{i=1}^{k} s_i,$$

however, is defined using the spin values $s_i(\sigma) := (-1)^{\sigma_i} \in \{\pm 1\}$.

The Farey spin chain has the following symmetries:

1. When one interprets $\{1,\ldots,k\}$ as a system of representatives of the residue class ring $\mathbb{Z}/k\mathbb{Z} = \{l + k\mathbb{Z} \mid l \in \mathbb{Z}\}$, then by cyclicity of the trace the energy function is invariant under the shift

$$\mathcal{S}_k : \mathbf{G}_k \to \mathbf{G}_k \quad , \quad \mathcal{S}_k(\sigma)_l := \sigma_{l-1} \tag{3}$$

on the configuration space $\mathbf{G}_k$ of the chain. So the interaction is translation-invariant, too ($J_k \circ \mathcal{S}_k = J_k$).

2. Since $AP = PB$ for $P := \binom{0\ 1}{1\ 0}$

$$M_k(\sigma_k,\ldots,\sigma_1) = PM_k(\sigma_1,\ldots,\sigma_k)^t P.$$

This implies the mirror symmetry

$$E_k(\sigma_k,\ldots,\sigma_1) = E_k(\sigma_1,\ldots,\sigma_k)$$

and a similar relation for the interaction coefficients.

3. Finally we notice that by 2) and transposition invariance of the trace

$$E_k(1-\sigma) = E_k(\sigma) \quad \text{for} \quad 1-\sigma := (1-\sigma_1,\ldots,1-\sigma_k)$$

so that

$$J_k(t) = 0 \quad \text{for} \quad \sum_{i=1}^{k} t_i \text{ odd.}$$

By 3) we need only consider t in the *even* subgroup

$$\mathbf{G}_k^e := \left\{ t \in \mathbf{G}_k \mid \textstyle\sum_{i=1}^k t_i \text{ even} \right\}.$$

$J_k(0) < 0$, since this is the negative mean of the (positive) energy function E_k. Note that this is the only interaction coefficient which does not influence the Gibbs measure.

3 A Polymer Model Interpretation

The notion of polymer models grew out of an abstraction of situations like the one encountered in the low temperature expansion of the Ising model. There one may decompose contours X into non-intersecting cycles γ_i ($X = (\gamma_1, \ldots, \gamma_l)$), and express the Boltzmann factor of the spin configuration in terms of products of activities attributed to these cycles (the activity $z(\gamma_i)$ of a cycle equals the exponential of its length, multiplied with minus the inverse temperature).

In an abstract setting (see, e.g., Gallavotti, Martin-Löf and Miracle-Solé [6], Glimm and Jaffe [7] and Simon [14]) one starts with a set P (which we assume here to be finite), whose elements are called *polymers*. Two given polymers $\gamma_1, \gamma_2 \in P$ may or may not overlap (be *incompatible*). Incompatibility is assumed to be a reflexive and symmetric relation on P.

Thus one may associate to a *l-polymer* $X := (\gamma_1, \ldots, \gamma_l) \in P^l$ an undirected graph $G(X) = (V(X), E(X))$ with vertex set $V(X) := \{1, \ldots, l\}$, vertices $i \neq j$ being connected by the edge $\{i, j\} \in E(X)$ if γ_i and γ_j are incompatible. Accordingly the l-polymer X is called *connected* if $G(X)$ is path-connected and *disconnected* if it has no edges ($E(X) = \emptyset$).

The corresponding subsets of P^l are called C^l resp. D^l, with $D^0 := P^0 := \{\emptyset\}$ consisting of a single element. Moreover $P^\infty := \bigcup_{l=0}^\infty P^l$ with the subsets $D^\infty := \bigcup_{l=0}^\infty D^l$ and $C^\infty := \bigcup_{l=1}^\infty C^l$. We write $|X| := l$ if $X \in P^l$.

Statistical weights or *activities* $z : P \to \mathbb{C}$ of the polymers are multiplied to give the activities $z^X := \prod_{i=1}^l z(\gamma_i)$ of l-polymers X. A system of statistical mechanics is called *polymer model* if its partition function Z has the form

$$Z = \sum_{X \in D_\Lambda^\infty} \frac{z^X}{|X|!}. \tag{4}$$

Then, up a normalization factor, the free energy is given by

$$\ln(Z) = \sum_{X \in C_\Lambda^\infty} \frac{n(X)}{|X|!} z^X, \tag{5}$$

with $n(X) := n_+(X) - n_-(X)$, $n_\pm(X)$ being the number of subgraphs of $G(X)$ connecting the vertices of $G(X)$ with an even resp. odd number of edges (see Gallavotti et al. [6]). It is known that (see, e.g. Prop. 20.3.5 of [7]) that

$$(-1)^{|V|-1} n(G) \geq 0. \tag{6}$$

This also follows from the *deletion-contraction property*

$$n(G) = n(G') - n(G''), \tag{7}$$

where G' is obtained from G by deleting an edge and G'' is the graph which arises by contracting the same edge of G (see, e.g. Read [12]).

In the present context of a chain with k spins we use

- the set P_k of $1 + k(k-1)$ polymers given by

  $$P_k := \{p\} \cup \{p_{l,r} \mid l \neq r \in \mathbb{Z}/k\mathbb{Z}\}.$$

- We map the polymers $\gamma \in P_k$ to group elements $\hat{\gamma} \in \mathbf{G}_k^e$ by setting $\hat{p} := 0$ and $\hat{p}_{l,r} := \delta_l + \delta_r$, where for $i \in \mathbb{Z}/k\mathbb{Z}$ the group element δ_i has the form $\delta_i(l) = 1$ if $l = i$ and zero otherwise.

  This map induces a map

  $$X = (\gamma_1, \ldots, \gamma_l) \mapsto \hat{X} := \sum_{i=1}^{l} \hat{\gamma}_i$$

  from the set P_k^∞ of multi-polymers to $\mathbf{G}_k^e$.

- The *support* of our polymers is given by $\mathrm{supp}(p) := \mathbb{Z}/k\mathbb{Z}$ and

  $$\mathrm{supp}(p_{l,r}) := \{l, l+1, \ldots, r-1, r\} \subset \mathbb{Z}/k\mathbb{Z}.$$

  Note that the polymer $p_{r,l}$ is different from $p_{l,r}$, although the group elements $\hat{p}_{l,r}$ and $\hat{p}_{r,l}$ coincide, and for $k = 2$ the supports $\mathrm{supp}(p) = \mathrm{supp}(p_{1,2}) = \mathrm{supp}(p_{2,1})$.

  The polymers γ and γ' are called overlapping or *incompatible* if

  $$\mathrm{supp}(\gamma) \cap \mathrm{supp}(\gamma') \neq \emptyset.$$

- We attribute to the polymers the activities

  $$z(p) := 3^{-|\mathrm{supp}(p)|} = 3^{-k} \quad \text{and} \quad z(p_{l,r}) := -3^{-|\mathrm{supp}(p_{l,r})|}. \tag{8}$$

Every group element $t \in \mathbf{G}_k^e$ allows for exactly two representations $t = \hat{X}$ by disjoint multi-polymers $X \in D_k^\infty$.

Lemma 1 *The Fourier transform $j_k := \mathcal{F}_k T_k$ can be written as*

$$j_k(t) = \left(\tfrac{3}{2}\right)^k \sum_{X \in D_k^\infty : \hat{X} = t} z(X) \qquad (t \in \mathbf{G}_k). \tag{9}$$

Proof.

- For t odd both sides are zero.
- For $t = 0$ we perform the sum to obtain

$$j_k(0) = 2^{-k} \sum_{\sigma \in \mathbf{G}_k} \mathrm{Trace}(M_k(\sigma)) = 2^{-k} \mathrm{Trace}(S^k)$$

  with $S := A + B = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, which has eigenvalues one and three. So $j_k(0) = (3^k + 1)/2^k$. On the other hand, the r.h.s of (9) is of the form

$$\left(\tfrac{3}{2}\right)^k (z(\emptyset) + z(p)) = \left(\tfrac{3}{2}\right)^k (1 + 3^{-k}).$$

- For $t \in \mathbf{G}_k^e \setminus \{0\}$ we assume without loss of generality, using cyclicity of the trace, that

$$t = (0_{m_1}, 1, 0_{m_2}, 1, \ldots, 1, 0_{m_{2n}}, 1)$$

  with $0_m = (0, \ldots, 0) \in \mathbf{G}_m$. Then with $D := A - B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

$$j_k(t) = 2^{-k} \mathrm{Trace}\left(S^{m_1} D S^{m_2} D \ldots D S^{m_{2n}} D\right).$$

  Now

$$D S^m D = S^m - (3^m + 1)\mathbb{1} \tag{10}$$

  commutes with S, and $D^2 = -\mathbb{1}$ so that

$$j_k(t) = (-1)^{n-1} 2^{-k} \mathrm{Trace}\left(S^{\Delta m_1} D S^{\Delta m_2} D\right)$$

with $\Delta m_1 := \sum_{l=1}^n m_{2l-1}$ and $\Delta m_2 := \sum_{l=1}^n m_{2l}$. Thus using (10) we arrive at

$$\begin{aligned} j_k(t) &= (-1)^{n-1} 2^{-k} \mathrm{Trace}\left(S^{\Delta m_1+\Delta m_2} - (1+3^{\Delta m_2}) S^{\Delta m_1}\right) \\ &= (-1)^n 2^{-k}\left(-(3^{\Delta m_1+\Delta m_2}+1) + (3^{\Delta m_1}+1)(3^{\Delta m_2}+1)\right) \\ &= (-1)^n 2^{-k}\left(3^{\Delta m_1} + 3^{\Delta m_2}\right). \end{aligned}$$

This equals the r.h.s. of (9). □

4 Comparison with the Number-Theoretical Spin Chain

The number-theoretical spin chain of length k has the *canonical* energy function

$$\mathbf{H}_k^C := \ln(\mathbf{h}_k^C) \quad \text{with} \quad \mathbf{h}_k^C : \mathbf{G}_k \to \mathbb{N}$$

inductively defined by

$$\mathbf{h}_0^C := 1, \quad \mathbf{h}_{k+1}^C(\sigma, \sigma_{k+1}) := \mathbf{h}_k^C(\sigma) + \sigma_{k+1}\mathbf{h}_k^C(1-\sigma), \qquad (\sigma \in \mathbf{G}_k). \tag{11}$$

It turns out to be useful to consider the *grand canonical* energy functions

$$\mathbf{H}_k^G : \mathbf{G}_k \to \mathbb{R} \quad \text{with} \quad \mathbf{H}_k^G(\sigma) := \mathbf{H}_{k+1}^C(\sigma, 1) \qquad (\sigma \in \mathbf{G}_k),$$

too, which is the logarithm of

$$\mathbf{h}_k^G : \mathbf{G}_k \to \mathbb{N} \quad , \quad \mathbf{h}_k^G(\sigma) := \mathbf{h}_{k+1}^C(\sigma, 1) = \mathbf{h}_k^C(\sigma) + \mathbf{h}_k^C(1-\sigma). \tag{12}$$

Namely in [8] polymer model techniques were applied to estimate the grand canonical interaction. $j_k^G := -\mathcal{F}_k \mathbf{H}_k^G$. These were applied to the subset

$$\tilde{P}_k := \{p_{l,r} \in P_k \mid l < r\} \tag{13}$$

of polymers (where the inequality $<$ in $\mathbb{Z}/k\mathbb{Z}$ is understood as the one for the representatives in $\{1, \ldots, k\}$). This is the set of polymers which contribute to the thermodynamic limit.

For $t \in \mathbf{G}_k \setminus \{0\}$ the resulting formula

$$j_k^G(t) = -\delta_{t,0} \cdot (\ln(2) + k\ln(3/2)) - \sum_{\substack{X \in \tilde{C}_k^\infty \\ \hat{X}=t}} \frac{n(X)}{|X|!} z^X \tag{14}$$

for these grand canonical interaction coefficients contains only nonnegative terms. This follows from (6) and the fact that all activities (8) of polymers in $\tilde{P}_k$ are negative. Similarly the canonical interaction of the number-theoretical spin chain was shown to be ferromagnetic.

Lemma 2 *The Farey interaction coefficients* $J_k(t) = -\mathcal{F}_k E_k(t)$ *can be written as*

$$J_k(t) = -\delta_{t,0} k \ln(3/2) - \sum_{\substack{X \in C_k^\infty \\ \hat{X}=t}} \frac{n(X)}{|X|!} z^X \qquad (t \in \mathbf{G}_k). \tag{15}$$

Proof. Since $j_k := \mathcal{F}_k T_k$, we have $E_k = 2^k \mathcal{F}_k j_k$ and

$$\begin{aligned} J_k(t) &= -2^{-k} \sum_{\sigma \in \mathbf{G}_k} E_k(\sigma) \cdot (-1)^{\sigma \cdot t} \\ &= -2^{-k} \sum_{\sigma \in \mathbf{G}_k} \ln \left[\sum_{s \in \mathbf{G}_k} j_k(s) \cdot (-1)^{s \cdot \sigma} \right] \cdot (-1)^{\sigma \cdot t} \\ &= -\delta_{t,0} \cdot k \ln(3/2) - 2^{-k} \sum_{\sigma \in \mathbf{G}_k} \ln \left[\sum_{X \in D_k^\infty} \frac{\tilde{z}_\sigma^X}{|X|!} \right] \cdot (-1)^{\sigma \cdot t} \end{aligned} \tag{16}$$

where the redefined single-polymer activities $\tilde{z}_\sigma(\gamma)$, $\gamma \in P_k$ are given by

$$\tilde{z}_\sigma(\gamma) := z_\sigma(\gamma) \cdot (-1)^{\sigma \cdot \hat{\gamma}},$$

that is $\tilde{z}_\sigma(p) = z(p)$ and $\tilde{z}_\sigma(p_{l,r}) = z(p_{l,r}) \cdot (-1)^{\sigma_l + \sigma_r}$. By (5) we get

$$\begin{aligned} & J_k(t) + \delta_{t,0} \cdot k \ln(3/2) \\ &= -2^{-k} \sum_{\sigma \in \mathbf{G}_k} \sum_{X \in C_k^\infty} \frac{n(X)}{|X|!} \tilde{z}_\sigma^X \cdot (-1)^{\sigma \cdot t} \\ &= -\sum_{X \in C_k^\infty} \frac{n(X)}{|X|!} z^X \cdot 2^{-k} \sum_{\sigma \in \mathbf{G}_k} (-1)^{\sigma \cdot (t + \hat{X})} = -\sum_{\substack{X \in C_k^\infty \\ \hat{X}=t}} \frac{n(X)}{|X|!} z^X, \end{aligned} \tag{17}$$

using the identity $\sum_{\sigma\in\mathbf{G}_k}(-1)^{\sigma\cdot s} = 2^k\delta_{s,0}$. $\square$

Although formula (15) looks very similar to (14), the sum is over all connected multipolymers based on the full set P_k of polymers, instead of the subset (13). Therefore not all terms in that sum are positive. By (6) and (8) the negative terms are precisely the ones containing an odd number of copies of the polymer p. Thus the positivity of the interaction for *finite* k does not follow immediately.

5 The Free Energy

Theorem 3 *The limit free energy density*

$$F(\beta) := \lim_{k\to\infty} F_k(\beta) \quad \textit{of} \quad F_k(\beta) := \frac{-1}{k\beta}\ln(Z_k(\beta)) \qquad (\beta > 0)$$

with k-spin partition function $Z_k(\beta) := \sum_{\sigma\in\mathbf{G}_k}\exp(-\beta E_k(\sigma))$ exists and equals the one of the number-theoretical spin chain.

Proof. We use the canonical and grand canonical ensembles as bounds for F_k.
1) Since the entries of the matrices $M_k(\sigma)$ are non-negative, an upper bound for $T_k = \mathrm{Trace}(M_k)$ is given by $\mathrm{Trace}\left(\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)M_k\right)$. But for $\sigma\in\mathbf{G}_k$

$$\mathrm{Trace}\left(\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)M_k(\sigma)\right) = \mathbf{h}_k^C(\sigma) \quad \text{and} \quad \mathrm{Trace}\left(\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)M_k(\sigma)\right) = \mathbf{h}_k^C(1-\sigma), \tag{18}$$

since both sides equal one for $k=0$, and

$$\begin{aligned}
&\mathrm{Trace}\left(\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)M_k(\sigma)\right)\\
&= \mathrm{Trace}\left(\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)A^{1-\sigma_k}B^{\sigma_k}M_{k-1}(\sigma_1,\ldots,\sigma_{k-1})\right)\\
&= \mathrm{Trace}\left(\left(\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)+\sigma_k\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)\right)M_{k-1}(\sigma_1,\ldots,\sigma_{k-1})\right)\\
&= \mathbf{h}_{k-1}^C(\sigma_1,\ldots,\sigma_{k-1}) + \sigma_k\mathbf{h}_{k-1}^C(1-\sigma_1,\ldots,1-\sigma_{k-1}) = \mathbf{h}_k^C(\sigma)
\end{aligned}$$

and similar for the second identity in (18). Adding these identities and, using Def. (12), shows that

$$T_k \le \mathbf{h}_k^G \qquad (k\in\mathbb{N}_0).$$

2) To derive a lower bound for T_k, we notice that for $\sigma\in\mathbf{G}_{k-1}$

$$\mathrm{Trace}\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)M_k(0,\sigma)\right) = \mathbf{h}_{k-1}^C(\sigma) \text{ and } \mathrm{Trace}\left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)M_k(0,\sigma)\right) = \mathbf{h}_{k-1}^C(1-\sigma),$$

since both sides equal one for $k = 1$, and for $\sigma \in \mathbf{G}_{k-1}$

$$\begin{aligned}
&\mathrm{Trace}\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)M_k(0,\sigma)\right)\\
&= \mathrm{Trace}\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)A^{1-\sigma_{k-1}}B^{\sigma_{k-1}}M_{k-1}(0,\sigma_1,\ldots,\sigma_{k-2})\right)\\
&= \mathrm{Trace}\left(\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)+\sigma_{k-1}\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\right)M_{k-1}(0,\sigma_1,\ldots,\sigma_{k-2})\right)\\
&= \mathbf{h}^C_{k-2}(\sigma_1,\ldots,\sigma_{k-2})+\sigma_{k-1}\mathbf{h}^C_{k-2}(1-\sigma_1,\ldots,1-\sigma_{k-2}) = \mathbf{h}^C_{k-1}(\sigma)
\end{aligned}$$

and similar for the second identity. Thus

$$T_k(0,\sigma) = T_k(1, 1-\sigma) \geq \mathbf{h}^C_{k-1}(\sigma) \qquad (k \in \mathbb{N}). \tag{19}$$

3) Since the (grand) canonical free energies are given by

$$F_k^{C/G}(\beta) = -\frac{\ln\left(\sum_{\sigma\in\mathbf{G}_k}\left(\mathbf{h}_k^{C/G}(\sigma)\right)^{-\beta}\right)}{\beta k},$$

these two inequalities imply

$$\frac{k-1}{k}F_k^C(\beta) - \frac{\ln 2}{\beta k} \leq F_k(\beta) \leq F_k^G(\beta).$$

The canonical and grand canonical ensembles have the same limit free energy, since

$$F_k^C \leq F_k^G \leq F_k^C + \frac{\ln(k+2)}{k}. \tag{20}$$

So the limit free energy F of the Farey chain coincides with the one of the number-theoretical spin chain

The lower inequality in (20) follows from (12), the upper inequality from $\mathbf{h}_k^G \leq (k+2)\cdot\mathbf{h}_k^C$, which is a consequence of (12) and the relation

$$\mathbf{h}_k^C(1-\sigma) \leq (k+1)\cdot\mathbf{h}_k^C(\sigma) \qquad (\sigma\in\mathbf{G}_k)$$

(which follows from Def. (11) by induction). □

The same conclusion was reached in [9] by a different method.

Corollary 4 *The Farey spin chain has exactly one phase transition, at $\beta = 2$.*

Proof. This follows from the corresponding statement in [3] for the number-theoretical spin chain. □

6 Low Temperature Magnetization

Due to the invariance of the energy function E_k w.r.t. spin flips, the mean magnetization m_k has expectation zero. However, the long-distance correlations are measured by the square of that variable.

Theorem 5 *In the low temperature phase* $\beta > 2$

$$\lim_{k\to\infty} \left\langle m_k^2 \right\rangle_k (\beta) = 1.$$

Proof. We complement estimate (19) by

$$T_k(\sigma) > k \qquad (\sigma \in \mathbf{G}_k, (0,\ldots,0) \neq \sigma \neq (1,\ldots,1)),$$

which follows inductively from Def. (1) by noticing that for such σ both off-diagonal entries are ≥ 1. We thus have

$$0 \leq Z_k(\beta) - 2 \cdot 2^{-\beta} \leq \sum_{n=1}^{\infty} a_k(n) n^{-\beta} \tag{21}$$

with

$$a_k(n) := |\{\sigma \in \mathbf{G}_k \mid \max(\mathbf{h}_k^C(\sigma), k+1) = n\}|.$$

It is known [10] that

$$Z_k^c(\beta) := \sum_{\sigma \in \mathbf{G}_k} \mathbf{h}_k^C(\sigma)^{-\beta}$$

can be written in the form

$$Z_k^c(\beta) = \sum_{n=1}^{\infty} \varphi_k(n) n^{-\beta}$$

with $\varphi_k(n) \leq \varphi(n)$ and $\varphi_k(n) = \varphi(n)$ for $n \leq k+1$,

$$\varphi(n) := |\{i \in \{1,\ldots,n\} \mid \gcd(i,n) = 1\}|$$

being Euler's φ-function. Thus for $\mathrm{Re}(\beta) > 1$

$$\lim_{k\to\infty} Z_k^c(\beta) = \sum_{n=1}^{\infty} \varphi(n) n^{-\beta} = \frac{\zeta(\beta - 1)}{\zeta(\beta)}.$$

Substituting the identity

$$\sum_{n=1}^{\infty} a_k(n)n^{-\beta} = Z_k^c(\beta) - \sum_{n=1}^{k} \varphi(n)\left(n^{-\beta} - (k+1)^{-\beta}\right)$$

for the r.h.s. of (21), we thus get

$$\begin{aligned}\lim_{k\to\infty} |Z_k(\beta) - 2\cdot 2^{-\beta}| &\le \lim_{k\to\infty} |(k+1)^{-\mathrm{Re}(\beta)}| \cdot \sum_{n=1}^{k} \varphi(n) \\ &\le \lim_{k\to\infty} |(k+1)^{-\mathrm{Re}(\beta)}| \cdot \sum_{n=1}^{k} n = 0,\end{aligned}$$

so that $\lim_{k\to\infty} Z_k(\beta) = 2\cdot 2^{-\beta}$ for $\mathrm{Re}(\beta) > 2$. Using $0 \le m_k^2 \le 1$ and $m_k^2((0,\ldots,0)) = m_k^2((1,\ldots,1)) = 1$, we thus get $\lim_{k\to\infty} \langle m_k^2\rangle_k(\beta) = 1$. $\square$

This extends the same conclusion, reached by a different argument for $\beta > 3$ [9].

For the ferromagnetic spin chain the limit mean magnetization $\langle m\rangle := \lim_{k\to\infty}\langle m_k\rangle_k$ equals 1 for the canonical ensemble and $\beta > 2$, whereas it vanishes in the high temperature region [3].

For the grand canonical ensemble, as for the Farey ensemble, $\langle m\rangle$ vanishes identically, since the interaction is even. Of course this does not say much about the actual structure of the extremal Gibbs states.

7 High Temperature Demagnetization

Now we consider the mean magnetization in the high temperature regime $\beta < 2$. To show that the expectation of the square vanishes, we need a correlation inequality. So consider for $n \in \mathbb{N}$ the configuration $\tau \in \mathbf{G}_{n+2}$ with spins $\tau_1 := \tau_{n+2} := 0$, $\tau_l := 1$ for $2 \le l \le n+1$, and the event

$$\mathcal{E}_k^n := \{\sigma \in \mathbf{G}_{k+n+2} \mid \sigma_l = \tau_l \text{ for } 1 \le l \le n+2\}$$

of an initial string of n adjacent 1-spins enclosed by 0-spins.

Due to the long range character of the interaction, one might think that, given $\mathcal{E}_k^n$, the ferromagnetic interaction would tend to align the other spins in the 1-direction (equal to $\tau_2 = \ldots = \tau_{n+1}$), at least if n is large.

This would mean a *negative* conditional expectation of $s_i = (-1)^{\tau_i}$ for $i \in \{n+3, \dots, n+k+2\}$. Because of the dominance of the multi-body interactions this is, however, not the case.

In fact, the non-inverted spins $(-1)^{\tau_1} = (-1)^{\tau_{n+2}} = 1$ tend to produce an anti-ferromagnetic effective coupling between the spins in the regions $2, \dots, n+1$ and $n+3, \dots, n+k+2$ they separate:

Proposition 6 *For $\Lambda \subset \{n+3, \dots, n+k+2\}$ and $\beta \geq 0$*

$$\langle s_\Lambda | \, \mathcal{E}_k^n \rangle_{k+n+2} (\beta) \geq 0 \quad \textit{with} \quad s_\Lambda := \prod_{i \in \Lambda} s_i, \tag{22}$$

$\langle f \, | \mathcal{E} \rangle_l$ denoting the expectation of $f : \mathbf{G}_l \to \mathbb{R}$, conditioned by the event $\mathcal{E}$.

Proof. We set

$$T_k^n : \mathbf{G}_k \to \mathbb{N} \quad , \quad T_k^n(\sigma) := T_{k+n+2}(\tau, \sigma)$$

and $E_k^n := \ln(T_k^n)$. We first prove

$$(\mathcal{F}_k E_k^n)(t) \leq 0 \qquad (t \in \mathbf{G}_k \setminus \{0\}), \tag{23}$$

using a polymer technique similar to the one above. Namely we redefine the set P_k of polymers by

$$P_k := \{p_m^L, p_m^R\}_{1 \leq m \leq k} \,\dot{\cup}\, \{p_{l,r}\}_{1 \leq l < r \leq k} \, ,$$

and map them to the group elements $\hat{p}_m^L := \hat{p}_m^R := \delta_m \in \mathbf{G}_k$ resp. $\hat{p}_{l,r} := \delta_l + \delta_r \in \mathbf{G}_k$. Depending upon the length n of the 1-substring, the polymer activities are given by

$$z(p_m^L) := -\frac{3^{-|\mathrm{supp}(p_m^L)|}}{2(n+1)} \, , \; z(p_m^R) := -\frac{3^{-|\mathrm{supp}(p_m^R)|}}{2(n+1)} \text{ and } z(p_{l,r}) := -3^{-|\mathrm{supp}(p_{l,r})|},$$

where

$$\mathrm{supp}(p_m^L) := \{1, \dots, m\}, \; \mathrm{supp}(p_m^R) := \{m, \dots, k\} \text{ and } \mathrm{supp}(p_{l,r}) := \{l, \dots, r\}.$$

Polymers with intersecting supports and the polymers p_m^L , p_m^R are mutually incompatible. We now claim that in analogy with (9) the Fourier transform $j_k^n := \mathcal{F}_k T_k^n$ can be written as

$$j_k^n(t) = 2(n+1) \left(\tfrac{3}{2}\right)^k \sum_{X \in D_k^\infty : \hat{X} = t} z(X) \qquad (t \in \mathbf{G}_k). \tag{24}$$

To show this, we write $t = (t_1, \ldots, t_k)$ uniquely in the form

$$t = (0_{m_1}, 1, 0_{m_2}, 1, \ldots, 0_{m_u}) \qquad (m_i \geq 0)$$

so that

$$j_k^n(t) = 2^{-k}\mathrm{Trace}\left(NS^{m_1}DS^{m_2}D \ldots S^{m_u}\right)$$

with $N := AB^nA = (n+1) \cdot \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) + D$.

- If u is odd, then

$$j_k^n(t) = (n+1) \cdot (-1)^{(u-3)/2} 2^{-k}\mathrm{Trace}\left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) S^{\Delta m_1} D S^{\Delta m_2} D\right)$$

  with $\Delta m_1 := \sum_{i=1}^{(u+1)/2} m_{2i-1}$ and $\Delta m_2 := \sum_{i=1}^{(u-1)/2} m_{2i}$. So

$$\begin{aligned} j_k^n(t) &= 2(n+1) \cdot (-1)^{(u-1)/2} 2^{-k} 3^{\Delta m_1} \\ &= 2(n+1) \cdot \left(\tfrac{3}{2}\right)^k \prod_{i=1}^{(u-1)/2} (-3^{-m_{2i}-2}). \end{aligned}$$

- If u is even, then

$$j_k^n(t) = (-1)^{u/2-1} 2^{-k}\mathrm{Trace}\left(DS^{\Delta m_1} DS^{\Delta m_2}\right)$$

  with $\Delta m_1 := \sum_{i=1}^{u/2} m_{2i-1}$ and $\Delta m_2 := \sum_{i=1}^{u/2} m_{2i}$. So using (10)

$$\begin{aligned} j_k^n(t) &= (-1)^{u/2} 2^{-k}(3^{\Delta m_1} + 3^{\Delta m_2}) \\ &= 2(n+1) \cdot \left(\tfrac{3}{2}\right)^k \cdot \left[\left(-\frac{3^{-m_u-1}}{2(n+1)}\right) \cdot \prod_{i=1}^{u/2-1} (-3^{-m_{2i}-2}) + \right. \\ &\qquad \left. \left(-\frac{3^{-m_1-1}}{2(n+1)}\right) \cdot \prod_{i=1}^{u/2-1} (-3^{-m_{2i+1}-2})\right]. \end{aligned}$$

In both cases this coincides with the r.h.s. of (24). Similar as in Lemma 2, we get

$$(\mathcal{F}_k E_k^n)(t) = \sum_{\substack{X \in C_k^\infty \\ \hat{X}=t}} \frac{n(X)}{|X|!} z^X \qquad (t \in \mathbf{G}_k \setminus \{0\}),$$

from which (23) follows, using (6) and the negativity of all polymer activities. Now

$$\langle s_\Lambda | \, \mathcal{E}_k^n \rangle_{k+n+2} (\beta) = \frac{\sum_{\sigma \in \mathbf{G}_k} s_\Lambda(\sigma) e^{-\beta E_k^n(\sigma)}}{\sum_{\sigma \in \mathbf{G}_k} e^{-\beta E_k^n(\sigma)}},$$

so that (22) is a consequence of the first GKS inequality for ferromagnets. □

Theorem 7 *In the high temperature phase* $0 \le \beta < 2$

$$\lim_{k \to \infty} \left\langle m_k^2 \right\rangle_k (\beta) = 0.$$

Proof. Since by translation invariance $\left\langle m_g^2 \right\rangle_g = \frac{1}{g} \sum_{j=1}^g \left\langle s_1 s_j \right\rangle_g$, it suffices to show that for $\varepsilon > 0$ there is a uniform correlation estimate of the form

$$|\left\langle s_1 s_j \right\rangle_g (\beta)| \le \varepsilon \qquad (j \in \{j_0(\varepsilon), \ldots, g - j_0(\varepsilon)\}). \tag{25}$$

We consider the family $\left\{ \mathcal{E}_{g-n-2}^{n,l} \right\}_{\substack{n=1,\ldots,n_{\max} \\ l=1,\ldots,n}}$ of events

$$\mathcal{E}_{g-n-2}^{n,l} := \mathcal{S}_k^{-l} \mathcal{E}_{g-n-2}^n \subset \mathbf{G}_g^1 \quad \text{with} \quad \mathbf{G}_g^j := \{\sigma \in \mathbf{G}_g \mid \sigma_j = 1\},$$

using the shift map (3) on $\mathbf{G}_g$. As these events are disjoint,

$$\sum_{n,l} \mathbb{P}_{\beta,g}(\mathcal{E}_{g-n-2}^{n,l}) = \mathbb{P}_{\beta,g} \left(\bigcup_{n,l} \mathcal{E}_{g-n-2}^{n,l} \right) \le \mathbb{P}_{\beta,g}(\mathbf{G}_g^1) = \tfrac{1}{2} \tag{26}$$

for the Gibbsian probability $\mathbb{P}_{\beta,k}(\mathcal{E}) := \sum_{\sigma \in \mathcal{E}} e^{-\beta E_k(\sigma)} / Z_k(\beta)$ of an event $\mathcal{E}$. On the other hand if $\beta < 2$, for $\varepsilon > 0$ there is a $n_{\max}(\varepsilon)$ with

$$\sum_{n=1}^{n_{\max}} \sum_{l=1}^{n} \mathbb{P}_{\beta,g}(\mathcal{E}_{g-n-2}^{n,l}) \ge \tfrac{1}{2}(1 - \varepsilon) \tag{27}$$

for all large g. This property, which is specific to the high-temperature region, can be proved as follows. We note that the thermodynamic limit of the internal energy

$$U := \lim_{k\to\infty} U_k \quad \text{with} \quad U_k := \left\langle \tfrac{1}{k} E_k \right\rangle_k$$

exists and equals $U(\beta) = \frac{d}{d\beta}\beta F(\beta)$. By concavity and analyticity of $\beta \mapsto \beta F(\beta)$, and by $F(\beta) = 0$ for $\beta \geq 2$ we conclude that

$$U(\beta) > 0 \qquad (\beta < 2).$$

This implies a positive limit density of spin flips between neighbouring spins and thus the existence of an $n_{\max}(\varepsilon)$ meeting (27).

Since by spin inversion symmetry

$$\langle s_1 s_j \rangle_g = -2 \left(\langle s_j \,|\, \mathcal{C}_g \rangle_g \cdot \mathbb{P}_{\beta,g}(\mathcal{C}_g) + \sum_{n,l} \left\langle s_j \,|\, \mathcal{E}^{n,l}_{g-n-2} \right\rangle_g \cdot \mathbb{P}_{\beta,g}(\mathcal{E}^{n,l}_{g-n-2}) \right)$$

for $\mathcal{C}_g := \mathbf{G}^1_g \setminus \cup_{n,l} \mathcal{E}^{n,l}_{g-n-2}$, by (26), (27) and Proposition 6

$$\langle s_1 s_j \rangle_g \leq 2\mathbb{P}_{\beta,g}(\mathcal{C}_g) \leq \varepsilon.$$

Together with a converse estimate this proves (25). $\square$

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